# Solutions

**Exam 3**
**Chapter 7**

**Name:**________________________

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Do not write your name on any other page. Answer the following questions. *Answers without proper evidence of knowledge will not be given credit.* Make sure to make reasonable simplifications. Do not approximate answers. Give exact answers. **Only scientific calculators are allowed on this exam.**

**Show your work!**

**1.** (10 points) Use Laplace transforms to solve the initial value problem

$$x'' + 3x' + 2x = 2; \quad x(0) = 2, x'(0) = -3.$$

$$\mathcal{L}\{x'' + 3x' + 2x\} = \mathcal{L}\{2\}$$

$$(s^2X(s) - 2s + 3) + 3(sX(s) - 2) + 2X(s) = \frac{2}{s}$$

$$(s^2 + 3s + 2)X(s) = \frac{2}{s} + 2s + 3 = \frac{2s^2 + 3s + 2}{s}$$

$$X(s) = \frac{2s^2 + 3s + 2}{s(s+2)(s+1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1} = \frac{A(s^2+3s+2) + B(s^2+s) + C(s^2+2s)}{s(s+2)(s+1)}$$

$$2 = A + B + C \Rightarrow 1 + B + C \Rightarrow B = 1 - C \Rightarrow B = 2$$
$$3 = 3A + B + 2C \Rightarrow 3 + B + 2C \Rightarrow 0 = (1 - C) + 2C \Rightarrow C = -1$$
$$2 = 2A \Rightarrow A = 1$$

$$X(s) = \frac{1}{s} + \frac{2}{s+2} - \frac{1}{s+1}$$

So $x(t) = \mathcal{L}^{-1}\{X(s)\} = 1 + 2e^{-2t} - e^{-t}.$

**2.** (10 points) Use Laplace transforms to solve the initial value problem

$$x'' + 8x' + 25x = 0; \quad x(0) = 2, x'(0) = 3.$$

$$\mathcal{L}\{0\} = \mathcal{L}\{x'' + 8x' + 25x\}$$

$$0 = (s^2X(s) - 2s - 3) + 8(sX(s) - 2) + 25X(s)$$

$$X(s)(s^2 + 8s + 25) = 2s + 19$$

$$X(s) = \frac{2s+19}{s^2+8s+25} = \frac{2s+19}{(s+4)^2+9} = \frac{2(s+4)}{(s+4)^2+9} + \frac{11}{(s+4)^2+9}$$

So $x(t) = \mathcal{L}^{-1}\{X(s)\} = e^{-4t}\left(2\cos 3t + \frac{11}{3}\sin 3t\right)$

**3.** (7 points) Use the fact that $\mathcal{L}\{(f*g)(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$ to find

$$\mathcal{L}^{-1}\{\frac{98}{(s-2)(s-3)}\}.$$

$$\frac{98}{(s-2)(s-3)} = 98 \cdot \frac{1}{s-2} \cdot \frac{1}{s-3} = 98 \cdot \mathcal{L}\{e^{2t}\}\mathcal{L}\{e^{3t}\}.$$

$$(e^{2t})*(e^{3t}) = \int_0^t e^{2(t-x)} e^{3x}\,dx = e^{2t}\int_0^t e^{x}\,dx = e^{2t} \cdot e^{x}\Big|_0^t$$

$$= e^{2t}(e^{t}-1)$$

$$= e^{3t} - e^{2t}.$$

Therefore

$$\mathcal{L}^{-1}\{\frac{98}{(s-2)(s-3)}\} = 98(e^{3t} - e^{2t}).$$

**4.** (7 points) Use the fact that $f(t) = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}$ to find

$$\mathcal{L}^{-1}\{\ln\left(1+\frac{1}{s^2}\right)\}.$$

$$F(s) = \ln\left(1+\frac{1}{s^2}\right) = \ln\left(\frac{s^2+1}{s^2}\right) = \ln(s^2+1) - \ln(s^2).$$

$$F'(s) = \frac{2s}{s^2+1} - \frac{2s}{s^2} = \frac{2s}{s^2+1} - \frac{2}{s}.$$

Thus $f(t) = \mathcal{L}^{-1}\{\ln(1+\frac{1}{s^2})\} = -\frac{1}{t}(2\cos t - 2)$

$$= \frac{2(1-\cos t)}{t}.$$

**5.** (7 points) Use the fact that $\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$ to find

$$\mathcal{L}\{f(t)\} \quad \text{where } f(t) = \begin{cases} \cos \pi t, & \text{if } 0 \le t \le 2\cancel{\pi} \\ 0, & \text{if } t > 2\cancel{\pi}. \end{cases}$$

$$f(t) = f(t)[1-u(t-2)]$$

$$= f(t) - u(t-2)f(t)$$

But $f(t-2) = \cos\pi(t-2) = \cos(\pi t - 2\pi) = \cos \pi t$.

Therefore

$$f(t) = f(t) - u(t-2)f(t-2).$$

So

$$\mathcal{L}\{f(t)\} = F(s) = \frac{s}{s^2+\pi^2} - \frac{e^{-2s} \cdot s}{s^2+\pi^2} = \frac{s(1-e^{-2s})}{s^2+\pi^2}.$$

**6.** Consider the differential equation

$$y^{(4)} + 2y'' + y = 4t\overset{3}{e}{}^{t}; \quad y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0.$$

(a) (6 points) Solve for the transform $Y(s) = \mathcal{L}\{y(t)\}$.
(Hint: You may need the formula $\{tf(t)\} = -F'(s)$.)

(b) (3 points) Find the general form of the partial fraction decomposition of $Y(s)$. You do not need to solve for the coefficients.

(a) $\mathcal{L}\{y^{(4)} + 2y'' + y\} = \mathcal{L}\{4te^{t}\}$

$$s^4Y(s) + 2s^2Y(s) + Y(s) = \frac{24}{(s-1)^4}$$

$$(s^4 + 2s^2 + 1)Y(s) = \frac{24}{(s-1)^4}$$

$$Y(s) = \frac{24}{(s-1)^4 \cdot (s^2+1)^2}$$

(b) $$Y(s) = \frac{A}{(s-1)^4} + \frac{B}{(s-1)^3} + \frac{C}{(s-1)^2} + \frac{D}{(s-1)} + \frac{Es+F}{(s^2+1)^2} + \frac{Gs+H}{s^2+1}$$